Level Curves/Contour Maps

Calculus III Project

Required Information

In this project we learn about the contour maps of various surfaces.

The cross-section of the surface $z = f(x, y)$ with a plane parallel to xy-plane, $z = k$, a constant, is a plane curve. The curve obtained for one value of $k$ is called a level curve. Every point on a given level curve is at the same height/depth on the surface from the xy-plane. A contour map (or contour diagram) consists of several level curves, $f(x, y) = k$, projected on the xy-plane.

As $k$ increases the height of the cross-section of $z = f(x, y)$ increases; thus the contour map gives an idea of the nature of the surface as the height or depth of the surface increases. Therefore it is easy to visualize a surface using its contour map.

If the surface is a plane parallel to the xy-plane, the contour map is the whole plane.

If the surface is a plane perpendicular to the xy-plane, the contour map is one line where the plane (if necessary by extending its graph) intersects the xy-plane.

If the surface is a plane inclined to the xy-plane, the cross-sections are lines for different values of $k$.

In fact, the contour map is a set of parallel lines.

If the surface is not a plane, the cross-sections are curves for different values of $k$. Hence the contour map is a set of curves.

If $k$ increases in equal increments, and the level curves are equidistant parallel lines then the surface is a plane; if the level curves are far apart then the surfaces rises or falls slowly; if the level curves are dense, then the surface rises or falls faster.

The contour maps are used by geologists, meteorologists, and surveyors. The mountains and oceans are mapped using the contour maps.

Solved Example

For each of the following equations: 1. Draw the surface. 2. Describe the contour map algebraically. 3. Draw the contour map. 4. Describe in words the level curves and how they are placed.

(a) $f(x, y) = 4$
(b) $f(x, y) = x + y$
(c) $f(x, y) = x^2 - y^2$
(d) $f(x, y) = e^{xy}$
(e) $f(x, y) = (x + y)/(x - y)$
(f) $f(x, y) = y - \cos(x)$
(g) \( f(x, y) = e^{\frac{1}{x^2 + y^2}} \)
with(plots):
Warning, the name changecoords has been redefined

Plotting \( z = f(x, y) \) can be done by plot3d command, giving ranges for \( x \) and \( y \). Note how we can title the diagram. Also when the surface is drawn you can change the axes and drawing using the menu for the plot. You can grab the surface at any point by the mouse and rotate the graph for a better view.

(a) \( f(x, y) = 4 \)

\[
\text{plot3d}(4, x = -5..5, y = -4..4, \text{title}=`z = 4`);
\]

\[
z = 4
\]

\[
\text{contourplot}(4, x = -5..5, y = -4..4, \text{title} = `\text{Contour Map for } z = 4`);
\]

\[
\text{Contour Map for } z = 4
\]

The contour map is the xy-plane.

(b) \( f(x, y) = x + y \)

\[
\text{plot3d}(x + y, x = -5..5, y = -4..4, \text{title}=`z = x + y`);
\]
\[ z = x + y \]

The contour map is a set of parallel lines: \( x + y = k \). For a particular level curve (line), the surface above the line neither goes up or down or in short, the surface is flat. Since the lines are parallel, the surface is flat everywhere. Thus the surface is a plane.

(c) \( f(x, y) = x^2 - y^2 \)

\[ > \text{plot3d}(x^2-y^2, x = -2..2, y = -2..2, \text{title}=\text{``z = x}^2 - y^2\text{''}); \]
The level curve \( x^2 - y^2 = k \) is a pair of lines when \( k = 0 \). For \( k \) positive, the level curve is a hyperbola that intersects the \( x \)-axis, and for \( k \) negative, the level curve is a hyperbola that intersects the \( y \)-axis.

This is seen in the following diagram.
(d) \( f(x, y) = \exp(x*y) \)

\[
> \text{plot3d}(\exp(x*y), x = -1..1, y = -1..1, \text{title} = `z = \exp(x*y)`); \\
\]

\[
> \text{contourplot}(\exp(x*y), x = -1..1, y = -1..1, \text{title} = `\text{Contour Map for } z = \exp(x*y)`); \\
\]
The level curves are \( \exp(xy) = k \) or \( xy = \ln k \). The curves are rectangular hyperbolas for \( k > 1 \).
For \( k = 1 \), the curve is \( xy = 0 \), i.e. the \( x \) and \( y \) axis. For \( 0 < k < 1 \), the curve is a hyperbola in the second and fourth quadrants.

When \( x, y \) are both positive or both negative, the curves are rectangular hyperbolas in the first and third quadrants. When \( x \) and \( y \) have different signs, the curves are in the second and fourth quadrant. For this surface, the height, \( z \), keeps increasing when both \( x, y \) have the same signs and their magnitude keeps increasing; the level curves are closer means the surface rises rapidly.

The curves are far apart in the second and fourth quadrants. In other words, when \( x \) and \( y \) have different signs, the surface rises slowly.

For illustration, the cross-section \( f(x, y) = 0.8 \) and the given surface is shown in the following, which can be rotated and inspected from different angles.

\[
\begin{align*}
\text{plot3d}\{\exp(x*y), 0.8\}, x = -1..1, y = -1..1, \text{title = `cross-section of } z = \exp(x*y), z = 0.8`};
\end{align*}
\]

\( f(x, y) = (x + y)/(x - y) \)

\[
\begin{align*}
\text{plot3d}\{(x + y)/(x - y), x = -1..1, y = -1..1, \text{title = `z = (x + y)/(x - y)`}}};
\end{align*}
\]
The contour map is given by

\[(x + y)/(x - y) = k\]
\[x + y = k(x - y)\]
\[(1 - k)x = -(1 + k)y\]
\[y = \frac{1 - k}{1 + k} x\]

There is no point on the contour map at \((0, 0)\) and along the line \(y = x\).

For \(k = 0\), the level curve is a line \(y = -x\) omitting the point \((0, 0)\).

For \(k = 1\), the line is \(y = 0\), the x-axis.

For \(k = -1\), the line is \(x = 0\), the y-axis.

For \(-1 < k < 1\), the level curve is a line with a negative slope, and lies in the second and fourth quadrants, not drawn by Maple.

For \(k > 1\), or \(k < -1\) the level curve is a line with a positive slope, and lies in the first and third quadrants.

There are sudden changes as \(x\) and \(y\) approach the line \(y = x\), i.e. the ratio, \(-\frac{1 - k}{1 + k}\), approaches 1 as \(k\) increases indefinitely.

The surface is cut up along plane \(y = x\). The surface is the xy-plane when \(y = -x\) away from \((0,\)
Clearer contour map can be seen by graphing the family of curves \( y = -\frac{1 - k}{1 + k} \) \( x \) for different values of \( k \). In the following the level curves for a few values of \( k \) between -1 and 2 have been drawn. We have changed the value of the constant in order to accommodate fractions.

\[
\begin{align*}
  &> s := \{ \text{seq}(y + (1 - m/4)/(1 + m/4)*x, \ m = -3..6) \}; \\
  &s := \{ y, y - \frac{1}{5} x, y + \frac{1}{3} x, y + \frac{5}{3} x, y + 3 x, y + \frac{1}{7} x, y + 7 x, y + \frac{3}{5} x, y + x, y
  \\
  &\quad - \frac{1}{9} x \} \\
  &> s := \{ y, y - 1/5*x, y + 1/3*x, y + 5/3*x, y + 3*x, y + 1/7*x, y + 7*x, y + 3/5*x, y + x, y - 1/9*x \};
\end{align*}
\]

\[
\begin{align*}
  &> \text{implicitplot}\{(y, y-1/5*x, y+1/3*x, y+5/3*x, y+3*x, y+1/7*x, y+7*x, y+3/5*x, y+x, y-1/9*x),\ x=-5..5,y=-5..5\};
\end{align*}
\]

\( f(x, y) = y - \cos x \)

\[
\begin{align*}
  &> \text{plot3d}(y - \cos(x), \ x = -2*Pi..2*Pi, \ y = -6..6, \ \text{title} = `z = y - \cos(x)`);\n\end{align*}
\]
\[ z = y - \cos(x) \]

The level curves are \( y = \cos(x) + k \).

The surface is a wavy surface.

Taking an intersection with a horizontal plane we get the following figure.

\[ > \text{contourplot}(y - \cos(x), x = -2\pi..2\pi, y = -6..6, \text{title} = \text{`Contour diagram for } z = y - \cos(x)\text{`}); \]

\[ > \text{plot3d}([y - \cos(x), 4], x = -2\pi..2\pi, y = -6..6, \text{title} = \text{`cross-section of } z = y - \cos(x)\text{ and } z = 4\text{`}); \]
\( (g) \ f(x, y) = \frac{1}{e^{x^2 + y^2}} \)

\[
\text{plot3d}(\exp(1/(x^2+y^2)), x = -3..3, y = -3..3, \text{title} = `z = \exp(1/(x^2+y^2))`);
\]

\[
z = \exp(1/(x^2+y^2))
\]

Since the surface rises very rapidly near the origin, the details cannot be seen in this graph. By restricting the height we can see more details on the surface. This is done in the following.

\[
> \text{implicitplot3d}(z = \exp(1/(x^2+y^2)), x = -4..4, y = -4..4, z = 0..10, \text{title} = `z = \exp(1/(x^2+y^2))`);
\]
\[ z = \exp\left(\frac{1}{x^2 + y^2}\right) \]

The surface rises rapidly for small values of \( x \) and \( y \), which the graph indicates.

> contourplot(\( \exp(1/(x^2+y^2)) \), x = -4..4, y = -4..4, title = `Contour`)

The contour map exists only for \( k > 0 \), actually only when \( k > 1 \), since

\[
\frac{1}{e^{x^2 + y^2}} = k
\]

gives rise to the equation

\[
\frac{1}{x^2 + y^2} = \ln k.
\]

or

\[
x^2 + y^2 = \frac{1}{\ln(k)}.
\]
This is a family of circles. In particular, when \( k = e \), \( x^2 + y^2 = 1 \) is the unit circle. This circle is seen on the surface, but not on the contour map drawn by Maple since it extends only up to \([-0.5, 0.5]\) on the x-axis, and up to \([-0.5, 0.5]\) on the y-axis. Because of the approximations made by Maple the level curves do not look like circles. We can draw the circles as follows.

\[
s := \{\text{seq}(x^2 + y^2 - 1/\ln(k), k = 2..6)\};
\]

\[
s := \left\{ x^2 + y^2 - \frac{1}{\ln(3)}, x^2 + y^2 - \frac{1}{\ln(4)}, x^2 + y^2 - \frac{1}{\ln(5)}, x^2 + y^2 - \frac{1}{\ln(6)}, \ldots \right\}
\]

\[
> \text{implicitplot}(s, x = -\sqrt{1/\ln(2)}..\sqrt{1/\ln(2)}, y = -\sqrt{1/\ln(2)}..\sqrt{1/\ln(2)});
\]

The level curves on this contour map come closer together as we approach the origin \((0, 0)\) on the plane. As \( k \) increases, \( 1/\ln(k) \) decreases and the radius of the circle decreases. This shows that the height of the surface rises rapidly as we approach the origin \((0, 0, 0)\) from any side.

**ASSIGNMENT**

**Problem**

For each of the following equations: 1. Draw the surface. 2. Describe the contour map algebraically. 3. Draw the contour map. 4. Describe in words the level curves and how they are placed.

(a) \( f(x, y) = xy \)
(b) \( f(x, y) = x^2 + y^2 \)
(c) \( f(x, y) = 3x - 3y \)
(d) \( f(x, y) = -x^2 - y^2 + 1 \)
(e) \( f(x, y) = y - x^2 \)
(f) \( f(x, y) = \cos \sqrt{x^2 + y^2} \)